Rutgers University: Algebra Written Qualifying Exam

August 2017: Problem 5 Solution

Exercise. Let $GL(n, F)$ denote the group of $n \times n$ invertible matrices with entries in the field F. Prove that $g_1, g_2 \in GL(n, \mathbb{Q})$ are conjugate in $GL(n, \mathbb{Q})$ if and only if they are conjugate in $GL(n, \mathbb{R})$.

Solution.

Two matrices g_1 and g_2 are **conjugate** IFF they are similar.
i.e. $\exists$a matrix P s.t. $g_1 = Pg_2P^{-1}$.

($\Longrightarrow$) is obvious.
If $g_1, g_2 \in GL(n, \mathbb{Q})$ are conjugate in $GL(n, \mathbb{Q})$, then $\exists\ P \in GL(n, \mathbb{Q}) \subset GL(n, \mathbb{R})$ s.t.

$$g_1 = Pg_2P^{-1}$$

Since $P \in GL(n, \mathbb{R})$, g_1 and g_2 are conjugate in $GL(n, \mathbb{R})$.

($\Longleftarrow$) Suppose g_1 and g_2 are conjugate in $GL(n, \mathbb{R})$.
Then they are similar in $GL(n, \mathbb{R})$ and
two matrices are similar IFF they have the same rational canonical form.
$\Longrightarrow$ g_1 and g_2 have the same rational canonical form over $GL(n, \mathbb{R})$.
$\Longrightarrow$ g_1 and g_2 have the same rational canonical form in $GL(n, \mathbb{Q}) \subset GL(n, \mathbb{R})$.
$\Longrightarrow$ g_1 and g_2 are conjugate in $GL(n, \mathbb{Q})$.

*See Dummit and Foote p. 472-479 (in particular 477)